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The N-wire exponential transmission line

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THE N -WIRE EXPONENTIAL

TRANSMISSION LINE

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ABSTRACT

This paper deals with the n-wire exponentially tapered transmission line. A characteristic impedance is defined, and the results of terminating in this impedance are investigated. A very long line and the condition of no reflections are considered, and the conclusions are compared.

Finally, three specific 3-wire line problems are worked with a characteristic impedance termination and a unit step input.

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TABLE OF CONTENTS

LIST OF FIGURES

THESIS PROLOGUE

This thesis is concerned with the analysis of the N-wire exponential transmission line. The analysis that follows is considered important because of the increased emphasis on multilayer, micro-electronic, and thin-film circuits which make use of uniform and non-uniform distributed parameter structures.

Figure 1 shows a pictorial representation of a tapered N-wire line. The exponential transmission line differs from its classical counterpart by the fact that the per unit impedances and admittances are not constant but vary as e^{2ax} and e^{-2ax}, respectively. The taper constant "a" is an arbitrary real constant. This variation can be accomplished by physical construction.

The distributed parameter approach to analysis is to subdivide the network lengthwise into elemental sections of length Δx. Figure 2 shows such a section located at position x. The functions $z_i(x)$ and $y_i(x)$ shown in figure 2 specify series line impedances per meter and shunt admittances per meter, respectively. A straightforward application of Kirchhoff's voltage and current laws followed by a limit as Δx approaches zero yields the following transformed Telegraphist's Equations:

$$\frac{\partial}{\partial x}\underline{V}(x,s) = \underline{Z}(x)\underline{I}(x,s) \tag{1}$$

$$\frac{\partial}{\partial x}\underline{I}(x,s) = \underline{Y}(x)\underline{V}(x,s) \tag{2}$$

where the above vectors and matrices are defined as follows:

$$\underline{V}(x,s) \equiv \mathrm{col}(V_i(x,s)) \tag{3}$$

$$\underline{I}(x,s) \equiv \mathrm{col}(I_i(s,s)) \tag{4}$$

$$\underline{Z}(x,s) \equiv \left[\, z_{ij}(x) \,\right] \tag{5}$$

$$\underline{Y}(x,s) \equiv \left[\, y_{ij}(x) \,\right] \qquad i,j = 1,2,\ldots\ldots,N \tag{6}$$

The above Telegraphists Equations (1) and (2) can be written in a more compact manner as

$$\frac{\partial}{\partial x}\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} = \begin{bmatrix} 0 & \underline{Z}(x) \\ \underline{Y}(x) & 0 \end{bmatrix}\begin{bmatrix} \underline{V}(x,s) \\ \underline{I}(x,s) \end{bmatrix} \tag{7}$$

The solutions to the Telegraphist's Equations are used as a basis for the analysis that follows with $|\underline{Z}(x)|$ and $|\underline{Y}(x)|$ being set proportional to e^{2ax} and e^{-2ax} respectively.

THE N-WIRE EXPONENTIAL TRANSMISSION LINE

I. Introduction

This thesis is concerned with the analysis of the N-wire exponential transmission line. The transfer matrix, which relates network input quantities to output quantities, of the general N-wire exponential line, was published recently by Bertnolli and Vandivort(1,2). This transfer matrix is used as the basis for this analysis.

The purpose of this thesis is to define a characteristic impedance and to find solutions for three specific 3-wire line transient analysis problems. The effect of terminating the N-wire line in its characteristic impedance is investigated.

II. Defining Characteristic Impedance

The solution of the Telegraphist's Equations with $\underline{Z}(x) = \underline{Z}\, e^{2ax}$ and $\underline{Y}(x) = \underline{Y}\, e^{-2ax}$, the matrices $\underline{Z}$ and $\underline{Y}$ being constant, is found in the following form:

$$\mathrm{col}(\underline{V}(x,s),\underline{I}(x,s)) = \underline{T}(x,s)\cdot\mathrm{col}(\underline{V}(0,s),\underline{I}(0,s)) \qquad (8)$$

where
$$\underline{V}(0) = \mathrm{col}(V_{21}(s),V_{22}(s),\ldots\ldots,V_{2n}(s)) \qquad (9)$$
$$\underline{I}(0) = \mathrm{col}(I_{21}(s),I_{22}(s),\ldots\ldots,I_{2n}(s)) \qquad (10)$$

are the variables at the receiving end of the line as shown in Figure 1.

The term $\underline{T}(x,s)$, called the transfer matrix, relates network input quantities to output quantities, and is given by the following equation:(1,2)

$$\underline{T}(x,s) = \begin{bmatrix} \underline{E}\, e^{ax} & 0 \\ 0 & \underline{E}\, e^{-ax} \end{bmatrix} \begin{bmatrix} \cosh(\Gamma x) - a\,\Gamma^{-1}\sinh(\Gamma x) & \underline{Z}\;\Gamma^{-T}\sinh(\Gamma^{T} x) \\ \underline{Y}\,\Gamma^{-1}\sinh(\Gamma x) & \cosh(\Gamma^{T} x) + a\,\Gamma^{-T}\sinh(\Gamma^{T} x) \end{bmatrix} \quad (11)$$

where $\underline{E}$ is the unit matrix, $\underline{\Gamma}^{2} = \underline{Z}\,\underline{Y} + a^{2}\underline{E}$, $\underline{\Gamma}^{-T} = (\underline{\Gamma}^{T})^{-1}$, and Γ^{T} represents the transpose of Γ.

It can be seen from the above equations that when the tapering constant "a" equals zero, the line reduces to the constant parameter case considered by Rice(3).

If the above matrix equations are rewritten in terms of exponential functions the following equations result:

$$\begin{aligned} \underline{V}(x) = {} & 1/2e^{ax}e^{\Gamma x}\left\{(\underline{E} - a\underline{\Gamma}^{-1})\underline{V}_2 + (\underline{E} + a\underline{\Gamma}^{-1})\underline{Z}_0\underline{I}_2\right\} \\ & + 1/2e^{ax}e^{-\Gamma x}\left\{(\underline{E} + a\underline{\Gamma}^{-1})\underline{V}_2 - (\underline{E} + a\underline{\Gamma}^{-1})\underline{Z}_0\underline{I}_2\right\} \end{aligned} \quad (12)$$

and

$$\begin{aligned} \underline{I}(x) = {} & 1/2e^{-ax}e^{\Gamma^{T} x}\left\{(\underline{E} - a\underline{\Gamma}^{-T})\underline{Z}_0^{-1}\underline{V}_2 + (\underline{E} + a\underline{\Gamma}^{-T})\underline{I}_2\right\} \\ & -1/2e^{-ax}e^{-\Gamma^{T} x}\left\{(\underline{E} - a\underline{\Gamma}^{-T})\underline{Z}_0^{-1}\underline{V}_2 - (\underline{E} - a\underline{\Gamma}^{-T})\underline{I}_2\right\} \end{aligned} \quad (13)$$

The impedance matrix $\underline{Z}_0 e^{2ax}$ is arbitrarily defined to be the line's characteristic impedance and is equal to

$$\underline{Z}_0 e^{2ax} = (\Gamma + a\underline{E})^{-1}\,\underline{Z}e^{2ax} \quad (14)$$

This expression for the characteristic impedance will be developed

III. Input Impedance of Very Long Line

Now consider a line of length d as shown in Figure 3. A simple change of variables will refer position to the sending end. Let $x = d - y$, where y is the distance from the sending end.

Rearranging the above equations gives

$$\begin{bmatrix} \underline{V}_2 \\ \underline{I}_2 \end{bmatrix} = \underline{T}^{-1}(y,s) \begin{bmatrix} \underline{V}_1(y) \\ \underline{I}_1(y) \end{bmatrix} \tag{15}$$

where $\underline{T}^{-1}(y,s)$ is the inverse matrix of $\underline{T}(y,s)$

As the length of the line d gets very large, $\underline{V}_2$ and $\underline{I}_2$ will approach zero. Making this approximation and considering the sending end by setting $y = 0$, results in the following equations:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = T^{-1}(0) \begin{bmatrix} \underline{V}_1(0) \\ \underline{I}_1(0) \end{bmatrix} \tag{16}$$

Carrying out the indicated multiplication gives

$$\underline{V}_1(0) = \underline{Y}^{-1} \left\{ \underline{\Gamma}^T \left[\sinh^{-1}(\underline{\Gamma}^T d)\right] \left[\cosh(\Gamma^T d)\right] - a\underline{E} \right\} e^{2ad}\, \underline{I}_1(0) \tag{17}$$

and since d is very large, the following approximation results

$$\underline{V}_1(0) \doteq \underline{Y}^{-1}(\Gamma^T - a\underline{E})e^{2ad}\underline{I}_1(0) \tag{18}$$

This equation indicates that the input impedance is

$$\underline{Z}_{in} \doteq \underline{Y}^{-1}(\Gamma^T - a\underline{E})e^{2ad} = \underline{Z}_o e^{2ad} \tag{19}$$

Therefore, the input impedance of a very long tapered line is equal to the characteristic impedance, and the characteristic impedance could be defined to be the input impedance of a very long line.

IV. Line Terminated in Characteristic Impedance

Figure 4 shows a line of length d, terminated at $x = 0$ in its characteristic impedance, $\underline{Z}_o e^{2ax}\Big|_{x=o} = \underline{Z}_o$.

Clearly the relationship $\underline{V}_2 = \underline{Z}_o \underline{I}_2$ holds and the following equation may be developed from (12)

$$\underline{V}(x) = e^{\Gamma x} e^{ax} \underline{V}_2 \tag{20}$$

Therefore, a characteristic impedance termination has eliminated the $e^{-\Gamma x}$ term.

In a similar manner from (13)

$$\underline{I}(x) = e^{\Gamma^T x} e^{-ax} \underline{Z}_o^{-1} \underline{V}_2 = e^{\Gamma^T x} e^{-ax} \underline{I}_2 \tag{21}$$

and from equations (20) and (21)

$$\underline{V}(x) = e^{\Gamma x} \underline{Z}_o e^{2ax} e^{-\Gamma^T x} \underline{I}(x) \tag{22}$$

Equation (22) can be shown to reduce to the form

$$\underline{V}(x) = \underline{Z}_o e^{2ax}\underline{I}(x) \tag{23}$$

This equation indicates that the voltage and current are related through $\underline{Z}_o\, e^{2ax}$ anywhere along the line.

V. Concepts of Traveling Waves

If the original solution to the exponential line problem equation (8) is rewritten in terms of exponentials the following equations will result:

$$\begin{aligned} e^{-ax}\underline{V}(x) = {} & 1/2e^{\Gamma x}\left\{(\underline{E} - a\Gamma^{-1})\underline{V}_2 + \Gamma^{-1}\underline{Z}\,\underline{I}_2\right\} \\ & + 1/2e^{-\Gamma x}\left\{(\underline{E} + a\Gamma^{-1})\underline{V}_2 - \Gamma^{-1}\underline{Z}\,\underline{I}_2\right\} \end{aligned} \tag{24}$$

$$\begin{aligned} e^{ax}\underline{I}(x) = {} & 1/2e^{\Gamma^{T} x}\left\{(\Gamma^{T})^{-1}\underline{Y}\,\underline{V}_2 + (\underline{E} + a(\Gamma^{T})^{-1}\underline{I}_2\right\} \\ & - 1/2e^{-\Gamma^{T} x}\left\{(\Gamma^{T})^{-1}\underline{Y}\,\underline{V}_2 - (\underline{E} - a(\Gamma^{T})^{-1}I_2\right\} \end{aligned} \tag{25}$$

If the exponential terms in equations (24) and (25) are considered to be traveling waves, the $e^{\Gamma x}$ term will travel from left to right on the line and $e^{-\Gamma x}$ term will travel from right to left. To eliminate reflections, set the coefficient of $e^{-\Gamma x}$ equal to zero. When this is done, the following relationship results:

$$\underline{V}_2 = (\Gamma + a\underline{E})^{-1}\,\underline{Z}\,\underline{I}_2 = \underline{Y}^{-1}\,(\Gamma^{T} - a\underline{E})\,\underline{I}_2$$

$$V_2 = Z\ \ I_2$$

It is, therefore, seen that in order to eliminate reflections, the line must be terminated in its characteristic impedance, and a stipulation of zero reflections could be used to define the characteristic impedance. This agrees with the previous section which concluded there was no $e^{-\Gamma x}$ term in the current or voltage equations when the termination was the characteristic impedance. It is therefore concluded that the concepts of traveling waves, a very long line, and characteristic termination are all three compatible in defining the characteristic impedance of a tapered line.

VI. Example 3-Wire Line Calculation

The most general case is considered first, and the circuit to be analyzed is that of Figure 5. In this case, the distributed capacitances and inductances are chosen to be $C_1 = C_2 = C_3 = 1$ farads/meter $L_1 = L_2 = L_3 = 1$ henrys/meter.
Therefore,

$$\underline{Z} = s\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{and} \qquad \underline{Y} = s\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{27}$$

Γ^2 is found using the values above as:

$$\Gamma^2 = \underline{Z}\,\underline{Y} + a^2\,\underline{E} = (a^2 + 3s^2)\,\underline{E}. \tag{28}$$

Since one input is an open circuit this forces I_{12} to equal zero. The voltage V_{11} is chosen to be a unit step or

1/s in the Laplace transform domain.

Since the network is terminated in its characteristic impedance the voltage and current are related through

$$\underline{V}(x) = \underline{Z}_o e^{2ax} \underline{I}(x) \tag{29}$$

Since $x = d$ at the sending end this gives

$$\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \underline{V}(d) = \underline{Z}_o e^{2ad} \begin{bmatrix} I_{11} \\ I_{12} \end{bmatrix} \tag{30}$$

Substituting the value of V_{11} and I_{12} and evaluating $\underline{Z}_o$ yields the following equation:

$$\begin{bmatrix} 1/s \\ V_{12} \end{bmatrix} = \frac{e^{2ad} s}{a + \sqrt{a^2 + 3s^2}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} I_{11} \\ 0 \end{bmatrix} \tag{31}$$

Simple algebra yields the solution:

$$I_{11} = e^{-2ad} \left(\frac{a + \sqrt{a^2 + 3s^2}}{2s^2} \right) \tag{32}$$

$$V_{12} = -1/2s \tag{33}$$

Attention is now turned to the receiving end, and the reader is referred to Figure 5. Again, because of the $\underline{Z}_o$ termination, the following equation can be written:

$$e^{-ad}\,\underline{V}(d) = 1/2e^{\underline{\Gamma} d}\left\{(\underline{E} - a\underline{\Gamma}^{-1})\underline{V}_2 + \underline{\Gamma}^{-1}\,\underline{Z}\,\underline{I}_2\right\} \tag{34}$$

and a straight forward substitution yields the following solution:

$$I_{22} = 0 \tag{35}$$

$$I_{21} = \frac{e^{-ad}(a+\sqrt{a^2 + 3s^2}\,)\;e}{2s^2}^{-\sqrt{a^2+3s^2}\,d} \tag{36}$$

Now that $\underline{I}_2 = \begin{bmatrix} I_{21} \\ I_{22} \end{bmatrix}$ has been found,

$\underline{V}_2$ can be found through the relationship:

$$\underline{V}_2 = \underline{Z}_o\,\underline{I}_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} \tag{37}$$

This gives the solutions:

$$V_{21} = e^{-ad}\,\frac{e^{-d\sqrt{a^2 + 3s^2}}}{s} \tag{38}$$

$$V_{22} = \frac{-V_{21}}{2} \tag{39}$$

The solutions in the time domain are (4,5):

$$I_{11}(t) = \frac{\sqrt{3}}{2} e^{-2ad}\left[e^{\frac{a}{\sqrt{3}} t} - \frac{a}{\sqrt{3}} \int_0^t e^{\frac{a}{\sqrt{3}}\sqrt{t^2-x^2}} J_1\left(\frac{a}{\sqrt{3}} x\right) dx \right] \tag{40}$$

$$V_{12}(t) = -1/2\ U(t) \tag{41}$$

$$I_{21}(t) = \frac{\sqrt{3}}{2} e^{-2ad}\left[e^{\frac{a}{\sqrt{3}} t} - \frac{a}{\sqrt{3}} \int_0^{\sqrt{t^2-3d^2}} e^{\frac{a}{\sqrt{3}}\sqrt{t^2-x^2}} J_1\left(\frac{a}{\sqrt{3}} x\right) \right] U(t-\sqrt{3}d) \tag{42}$$

$$V_{21}(t) = e^{-ad}\left[U(t-\sqrt{3}d) - \int_0^t \frac{ad}{\sqrt{x^2-3d^2}} J_1\left(\frac{a}{\sqrt{3}}\sqrt{x^2-3d^2}\right) U(x-\sqrt{3}d)\ dx \right] \tag{43}$$

where $J_1(x)$ in a Bessel function of the first kind and first order, and $U(t)$ is the step function.

For the case where $a = 0$ the following simplifications result:

$$I_{11} = \frac{\sqrt{3}}{2}\frac{1}{s} \qquad V_{21} = \frac{e^{-\sqrt{3}ds}}{s}$$

$$I_{21} = \frac{\sqrt{3}}{2}\frac{1}{s} e^{-\sqrt{3}ds} \qquad V_{22} = -\frac{1}{2} V_{21}$$

the other variables remaining the same. In the time domain

$$I_{11}(t) = \sqrt{3}/2\ U(t) \qquad V_{21}(t) = U(t-\sqrt{3}\ d)$$

$$I_{21}(t) = \sqrt{3}/2\ U(t-\sqrt{3}\ d) \qquad V_{22}(t) = -1/2\ U(t-\sqrt{3}\ d)$$

Since the time domain solutions could not be found in a closed finite form, a value of the tapering constant "a" must be chosen so the integration can be carried out numerically. A value of 1 is chosen for "a" and the plot of the responses is shown in figures 7,8,9,and 10.

With $a = 0$, the constant parameter case, the responses are shown in figures 11,12, and 13. There is a marked difference between the constant parameter and tapered current response, the tapered current response being unbounded. It should also be pointed out that the tapered line acts as a transformer. The tapered line introduced time delay as did the constant line case, but in the tapered case the load responses are different in magnitude from the source quantities as opposed to the constant magnitude in the constant line case. As seen from the analytic solutions, for very large time the source and load variables will be approximately equal.

Example 2

The above problem is repeated with the parameters changed to $C_1 = C_2 = 1$, $C_3 = 0$ farad/meter and $L_1 = L_3 = 1$, $L_2 = 0$ henrys/meter. The circuit is again terminated in its characteristic impedance.

With these new parameters the $\underline{Z}$ and $\underline{Y}$ matrices take on the values

$$\underline{Z} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{Y} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and Γ^2 is found using the above values as

$$\underline{\Gamma}^2 = \underline{Z}\,\underline{Y} + a^2\,\underline{E} = (s^2 + a^2)\,\underline{E} \tag{45}$$

Again, the characteristic impedance is found, and it turns out to be a particularly simple form.

$$\underline{Z}_o = (\underline{\Gamma} + a\underline{E})^{-1}\,\underline{Z} = \frac{s}{\sqrt{a^2 + s^2} + a}\,\underline{E} \tag{46}$$

and when the analysis is carried out in a procedure parallel to the previous example, the following solutions result:

$$I_{11} = e^{-2ad}\,\frac{\sqrt{s^2 + a^2} + a}{s^2} \tag{47}$$

$$V_{12} = V_{22} = 0 \tag{48}$$

$$I_{21} = e^{-ad}\,\frac{\sqrt{s^2 + a^2} + a}{s^2}\,e^{-d\sqrt{a^2 + s^2}} \tag{49}$$

$$I_{12} = I_{22} = 0 \tag{50}$$

$$V_{21} = e^{-ad}\,\frac{e^{-d\sqrt{a^2 + s^2}}}{s} \tag{51}$$

The solutions in the time domain are:

$$I_{11}(t) = e^{-2ad}\left[e^{at} - a\int_0^t e^{a\sqrt{t^2 - x^2}}\, J_1(ax)\, dx\right] \tag{52}$$

$$I_{21}(t) = e^{-2ad}\left[e^{at} - a\int_0^{\sqrt{t^2-d^2}} e^{a\sqrt{t^2 - x^2}}\, J_1(ax)\, dx\right] U(t-d) \tag{53}$$

$$V_{21}(t) = e^{-ad}\left[U(t-d) - \int_0^t \frac{ad}{(x^2 - d^2)^{1/2}} J_1\left(a\sqrt{x^2 - d^2}\right) U(x-d)\, dx\right] \tag{54}$$

Again a value of "a" must be chosen, $a = 1$, and the integration carried out numerically. The results are shown in figures 14, 15, and 16. The case for $a = 0$ is shown in figures 17 and 18. Notice should again be taken of the unbounded nature of the response currents for the $a = 1$ case, and of the transformer effect between load and source variables.

Now considering the case $a = 0$ gives the solutions:

$I_{11} = 1/s$

$V_{12} = V_{22} = 0$

$I_{21} = e^{-sd}\,\dfrac{1}{s}$

$I_{12} = I_{22} = 0$

$V_{22} = \dfrac{e^{-sd}}{s}$

$V_{21} = \dfrac{e^{-sd}}{s}$

Or equivalently in the time domain

$I_{11} = U(t)$

$I_{21}(t) = U(t - d)$

$V_{22}(t) = U(t - d)$

$V_{21}(t) = U(t - d)$

Example 3

This example represents a practical situation in which the parameters take on the values $C_1 = C_2 = 1$, $C_3 = 0$ farads/meter and $L_1 = L_2 = L_3 = 1$ henrys/meter.

The circuit is shown in figure 6, where it is again terminated in its characteristic impedance.

The characteristic impedance must again be found but it presents more of a problem this time since some of the matrices are not diagonal. To find $\underline{Z}_o$ Sylvester's Theorem (6) will be used. The eigenvalues of $\underline{\Gamma}^2$ are found as follows:

$$\begin{vmatrix} 2s^2 + a^2 - \lambda & -s^2 \\ -s^2 & 2s^2 + a^2 - \lambda \end{vmatrix} = 0 \qquad (55)$$

which gives the following solutions:

$$\lambda_1 = s^2 + a^2 \qquad \lambda_2 = 3s^2 + a^2 \tag{56}$$

In this special case Sylvester's Theorem may be written as

$$f(\underline{\Gamma}^2) = \sum_{i=1}^{2} f(\lambda_i) \frac{\prod_{j=1}^{2} (\underline{\Gamma}^2 - \lambda_j \underline{E})}{\prod_{\substack{j=1 \\ j \neq i}}^{2} (\lambda_i - \lambda_j)} \tag{57}$$

where $f(\underline{\Gamma}^2)$ is any function of $\underline{\Gamma}^2$ and λ_i represents the respective eigenvalue of $\underline{\Gamma}^2$. The characteristic impedance is expressed as a function of $\underline{\Gamma}^2$ in the following manner:

$$\underline{Z}_o = \underline{Y}^{-1} \left[(\underline{\Gamma}^2)^{1/2} - a\underline{E} \right] \tag{58}$$

Application of Sylvester's Theorem yields the following for $\underline{Z}_o$.

$$\underline{Z}_o = \begin{bmatrix} \dfrac{(\sqrt{a^2 + s^2} + \sqrt{a^2 + 3s^2} - 2a)}{2s} & \dfrac{(\sqrt{a^2 + s^2} - \sqrt{a^2 + 3s^2})}{2s} \\ \\ \dfrac{(\sqrt{a^2 + s^2} - \sqrt{a^2 + 3s^2})}{2s} & \dfrac{(\sqrt{a^2 + s^2} + \sqrt{a^2 + 3s^2} - 2a)}{2s} \end{bmatrix} \tag{59}$$

The circuit is again analyzed using the same procedure as used in the previous example. The sending end is analyzed first, and the following values are found:

$$I_{11} = \frac{2\, e^{-2ad}}{\sqrt{a^2 + s^2} + \sqrt{a^2 + 3s^2} - 2a} \tag{60}$$

$$V_{12} = \frac{\sqrt{a^2 + s^2} - \sqrt{a^2 + 3s^2}}{s\left(\sqrt{a^2 + s^2} \quad \sqrt{a^2 + 3s^2} - 2a\right)} \tag{61}$$

Again, going back to the original equation and using the relation $\underline{V}_2 = \underline{Z}_o \underline{I}_2$ we get:

$$e^{-ad}\, \underline{V}(d) = 1/2\, e^{\Gamma d} \left\{ (\underline{E} - a\Gamma^{-1})\, \underline{Z}_o + \Gamma^{-1} \underline{Z} \right\} \underline{I}_2 \tag{62}$$

Since we know the value of the matrix $\underline{V}(d)$, we need to find $\underline{I}_2$. This is found most easily by making use of Sylvester's Theorem to take the inverse of the quantity which is a function of Γ^2.

When this is carried out, the following solution results:

$$I_{21} = e^{ad} \frac{\left(e^{-d\sqrt{a^2 + s^2}} + e^{-d\sqrt{a^2 + 3s^2}}\right)}{\sqrt{a^2 + s^2} + \sqrt{a^2 + 3s^2} - 2a} \tag{63}$$

$$I_{22} = e^{-ad} \frac{e^{-d\sqrt{a^2+s^2}} - e^{-d\sqrt{a^2+3s^2}}}{\sqrt{a^2+s^2} + \sqrt{a^2+3s^2} \quad -2a} \tag{64}$$

The matrix $\underline{V}_2$ is again found using the relation

$$\underline{V}_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \underline{Z}_o \underline{I}_2 = \underline{Z}_o \begin{bmatrix} I_{21} \\ I_{22} \end{bmatrix} \tag{65}$$

which gives,

$$V_{21} = e^{-ad} \frac{(\sqrt{a^2+s^2} - a)\, e^{-d\sqrt{a^2+s^2}} + (\sqrt{a^2+3s^2} - a)\, e^{-d\sqrt{a^2+3s^2}}}{s(\sqrt{a^2+s^2} + \sqrt{a^2+3s^2} \quad - \quad 2a)}$$

and

$$V_{22} = e^{-ad} \frac{(\sqrt{a^2+s^2} - a)\, e^{-d\sqrt{a^2 - s^2}} \quad (\sqrt{a^2+3s^2} - a)\, e^{-d\sqrt{a^2+3s^2}}}{s(\sqrt{a^2+s^2} + \sqrt{a^2+2s^2} \quad -2a)}$$

If we again assume $a = 0$, this reduces to the constant parameter case. The above solutions then reduce to the following simplified forms:

$$I_{11} = \frac{2}{1+\sqrt{3}} \; \frac{1}{s}$$

$$V_{12} = (\sqrt{3} - 2)\, 1/s$$

$$I_{21} = \frac{1}{\quad} \; \frac{e^{-sd} + e^{-\sqrt{3}ds}}{\quad}$$

$$I_{22} = \frac{1}{1+\sqrt{3}} \frac{e^{-sd} - e^{-\sqrt{3}sd}}{s}$$

$$V_{21} = \frac{1}{1+\sqrt{3}} \frac{e^{-sd} + \sqrt{3}\, e^{-\sqrt{3}sd}}{s}$$

$$V_{22} = \frac{1}{1+\sqrt{3}} \frac{e^{-sd} - \sqrt{3}\, e^{-\sqrt{3}\, sd}}{s}$$

and the characteristic impedance $\underline{Z}_o$ is resistive,

$$\underline{Z}_o = 1/2 \begin{bmatrix} (1+\sqrt{3}) & (1-\sqrt{3}) \\ (1-\sqrt{3}) & (1+\sqrt{3}) \end{bmatrix} \qquad (68)$$

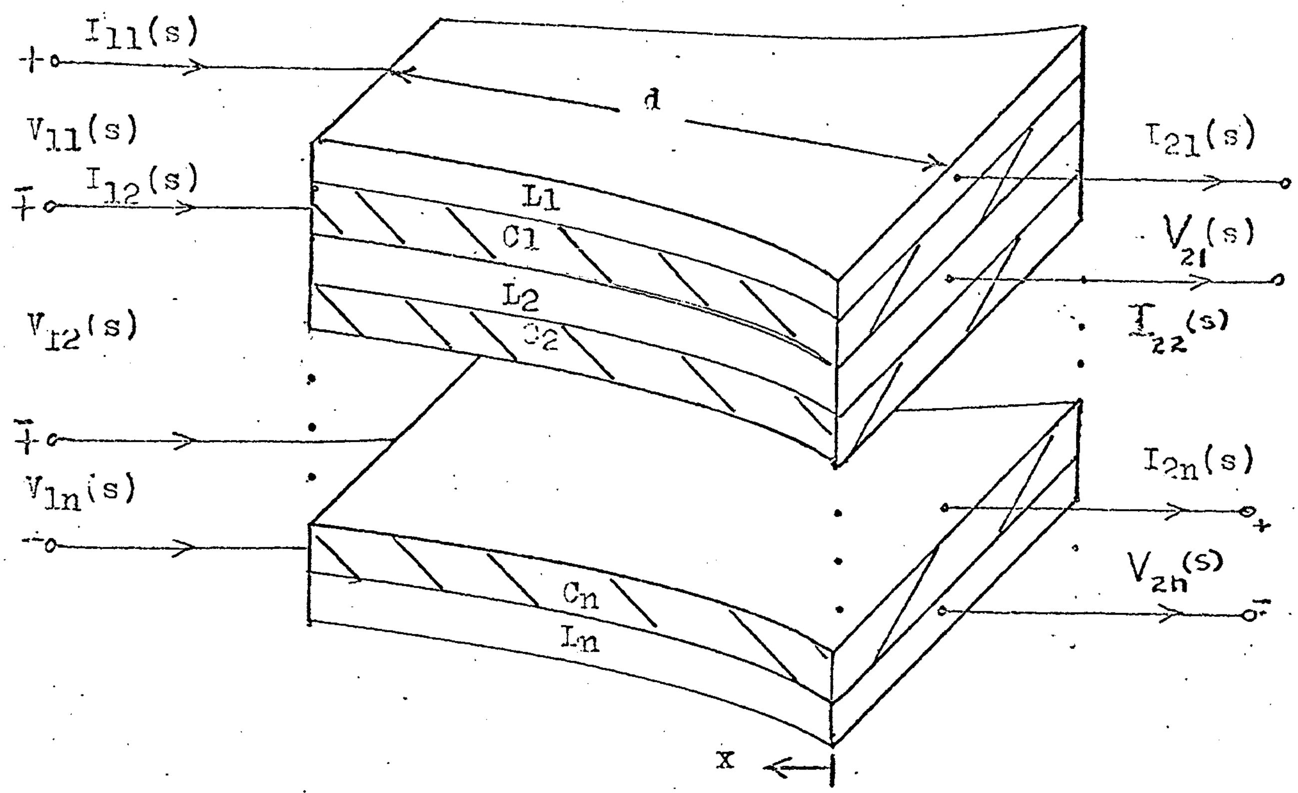
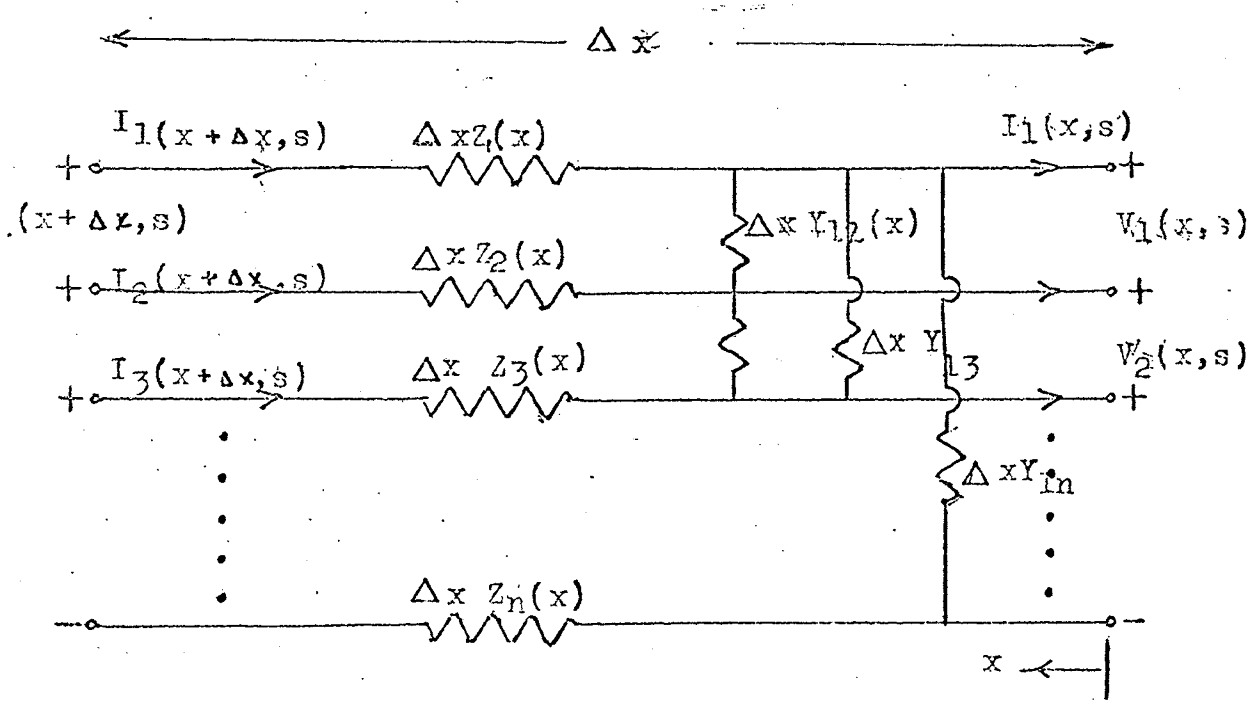


Figure 1. 2n+2 Terminal Tapered Line of Length "d".



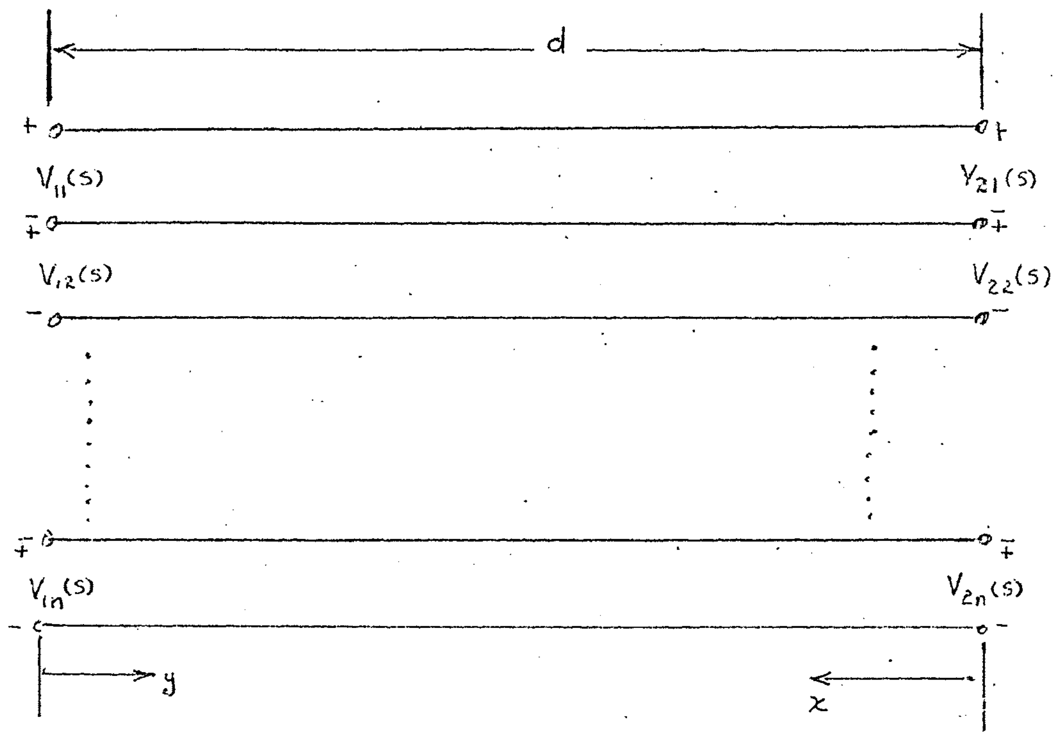


FIG. 3 2n+2 TERMINAL EXPONENTIAL LINE
OF LENGTH d.

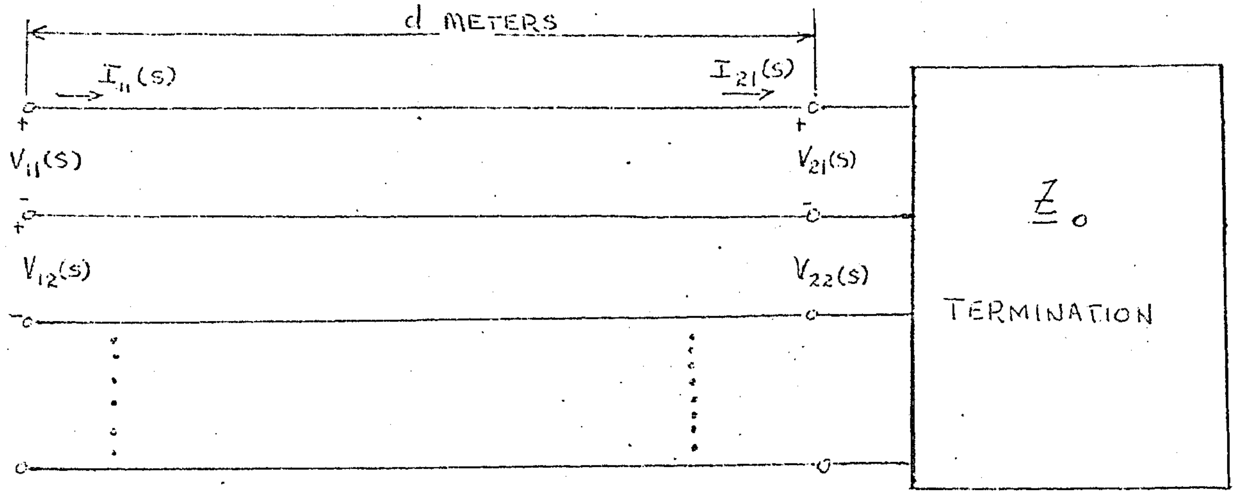


FIG. 4 LINE OF LENGTH d WITH Z_0 TERMINATION

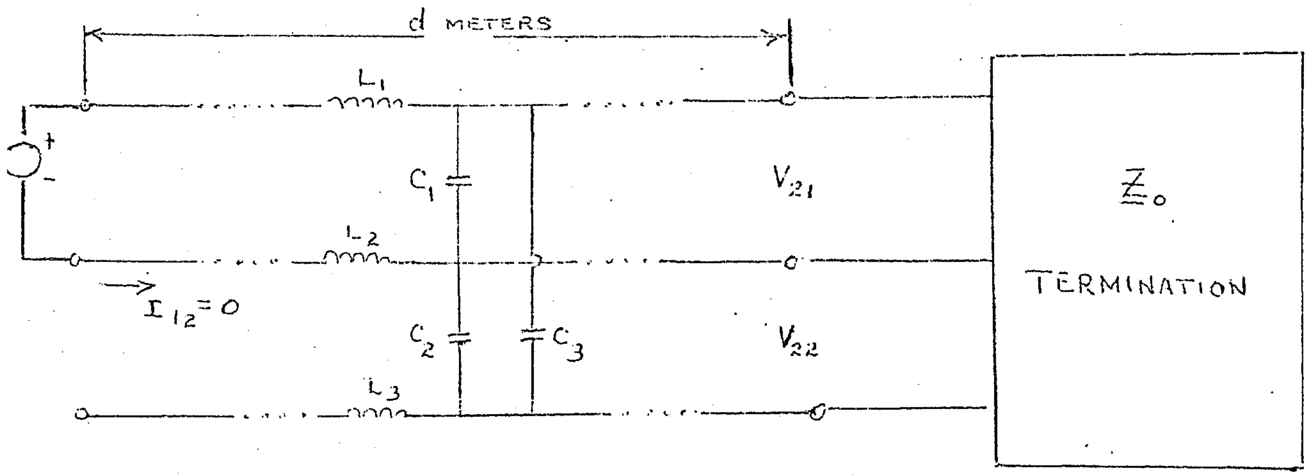


FIG. 5 SIX TERMINAL LINE SHOWING CAPACITANCE AND INDUCTANCE OF EXAMPLE 1.

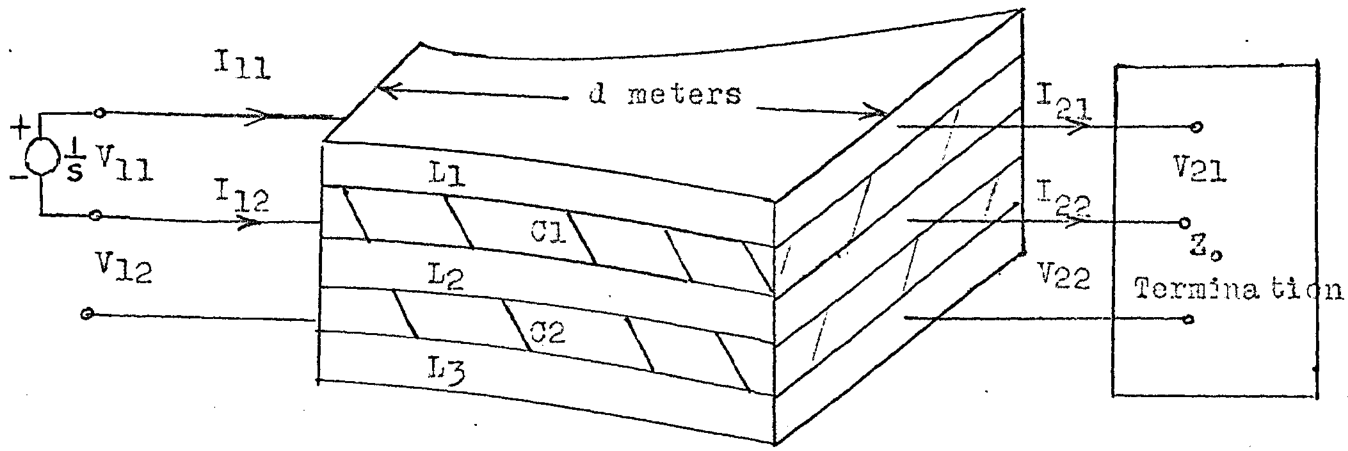


Figure 6. Six Terminal LC Line With $C_3 = 0$.

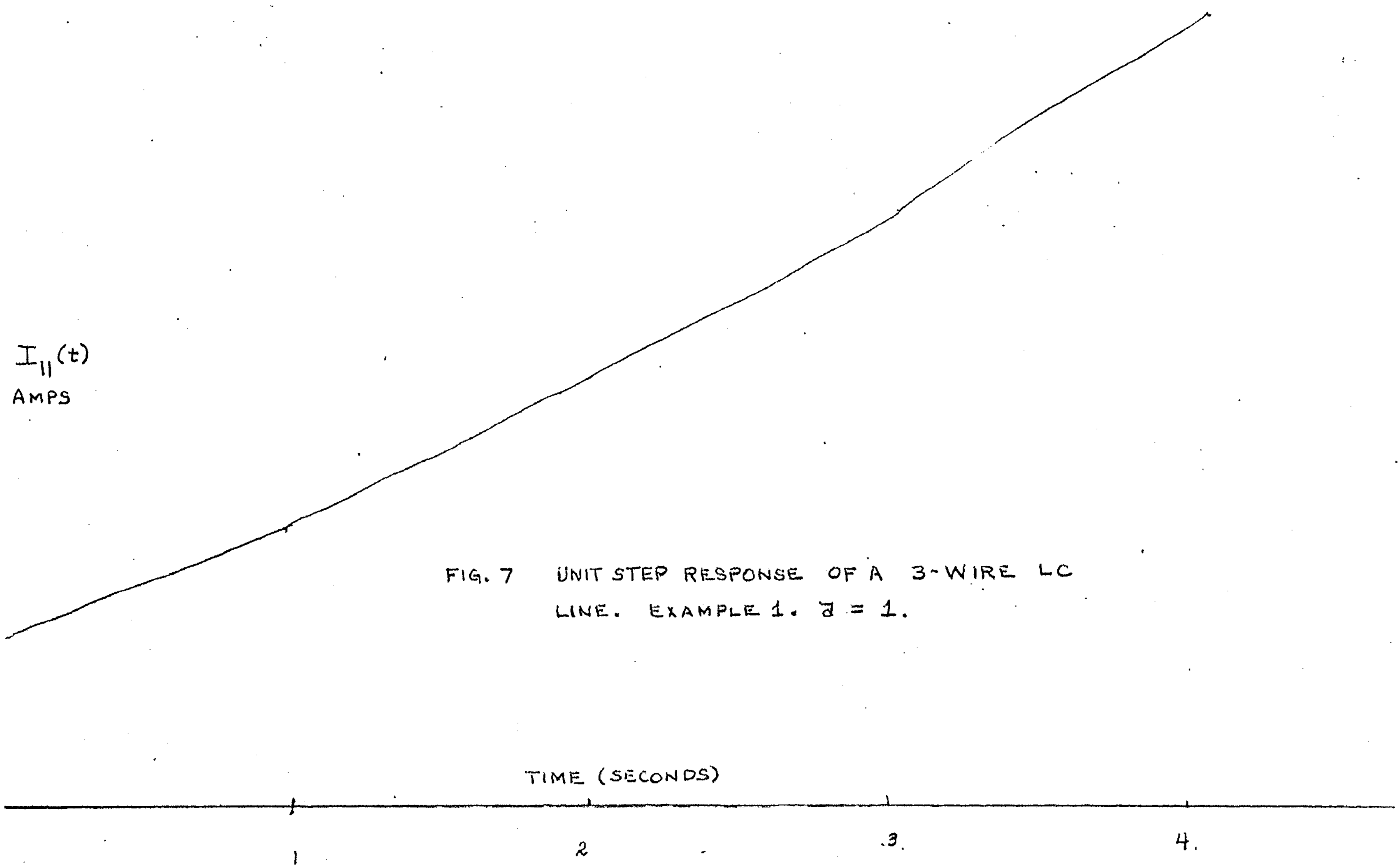


FIG. 7 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 1. $\partial = 1$.

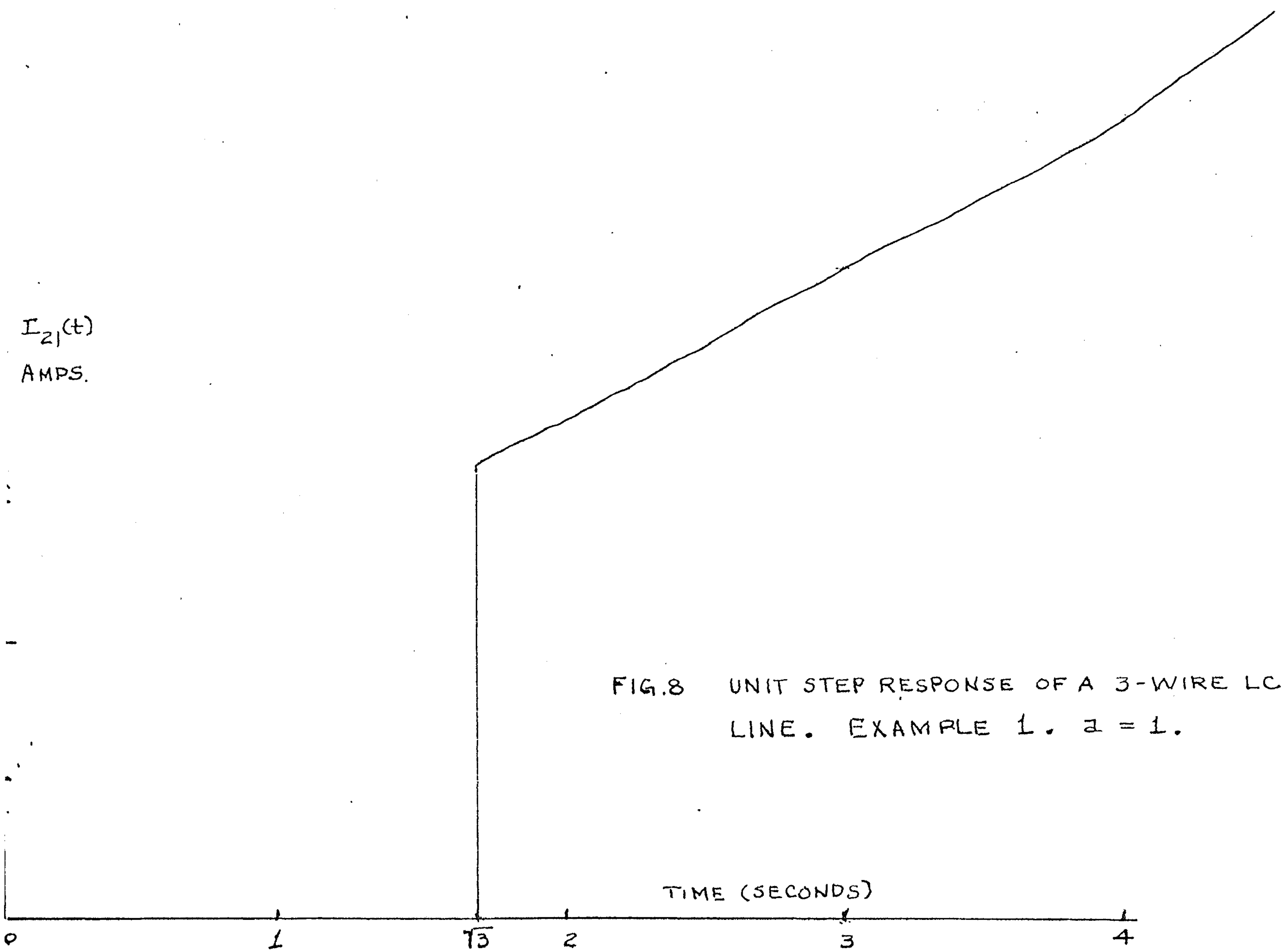


FIG.8 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 1. $a = 1$.

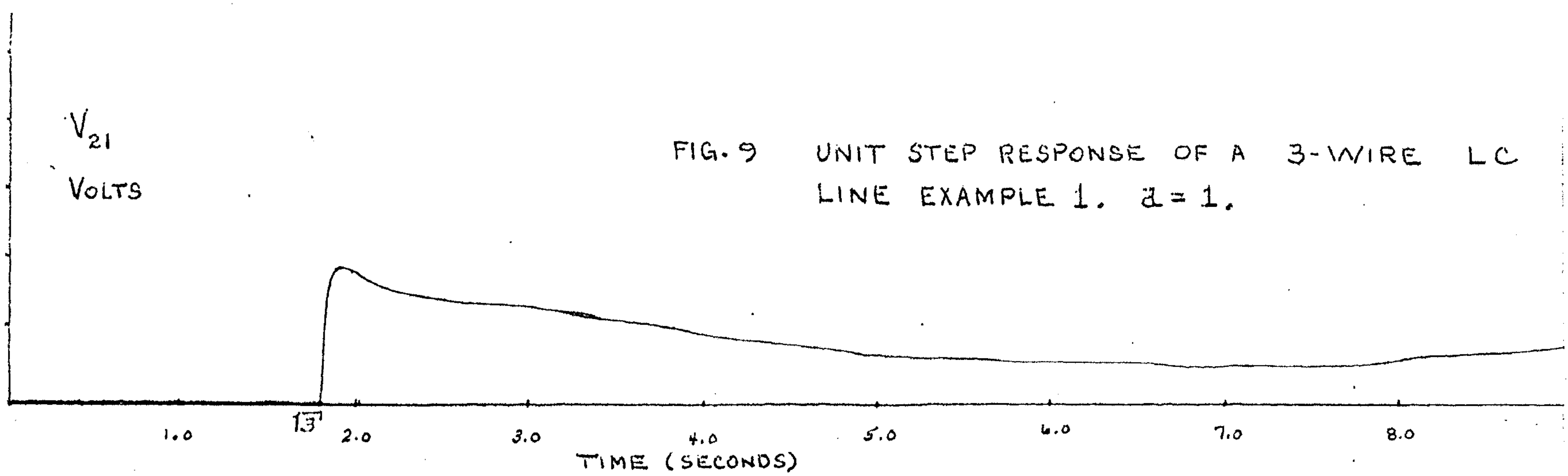


FIG. 9 UNIT STEP RESPONSE OF A 3-WIRE LC LINE EXAMPLE 1. $\alpha = 1$.

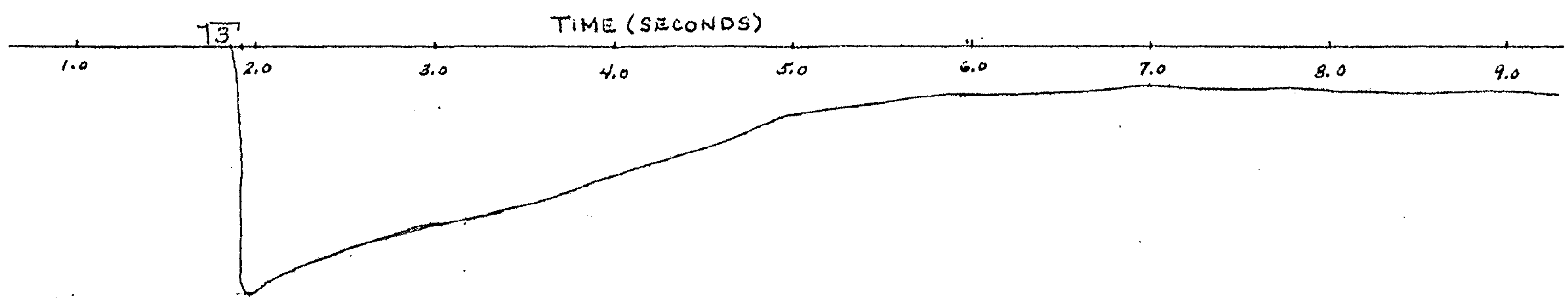
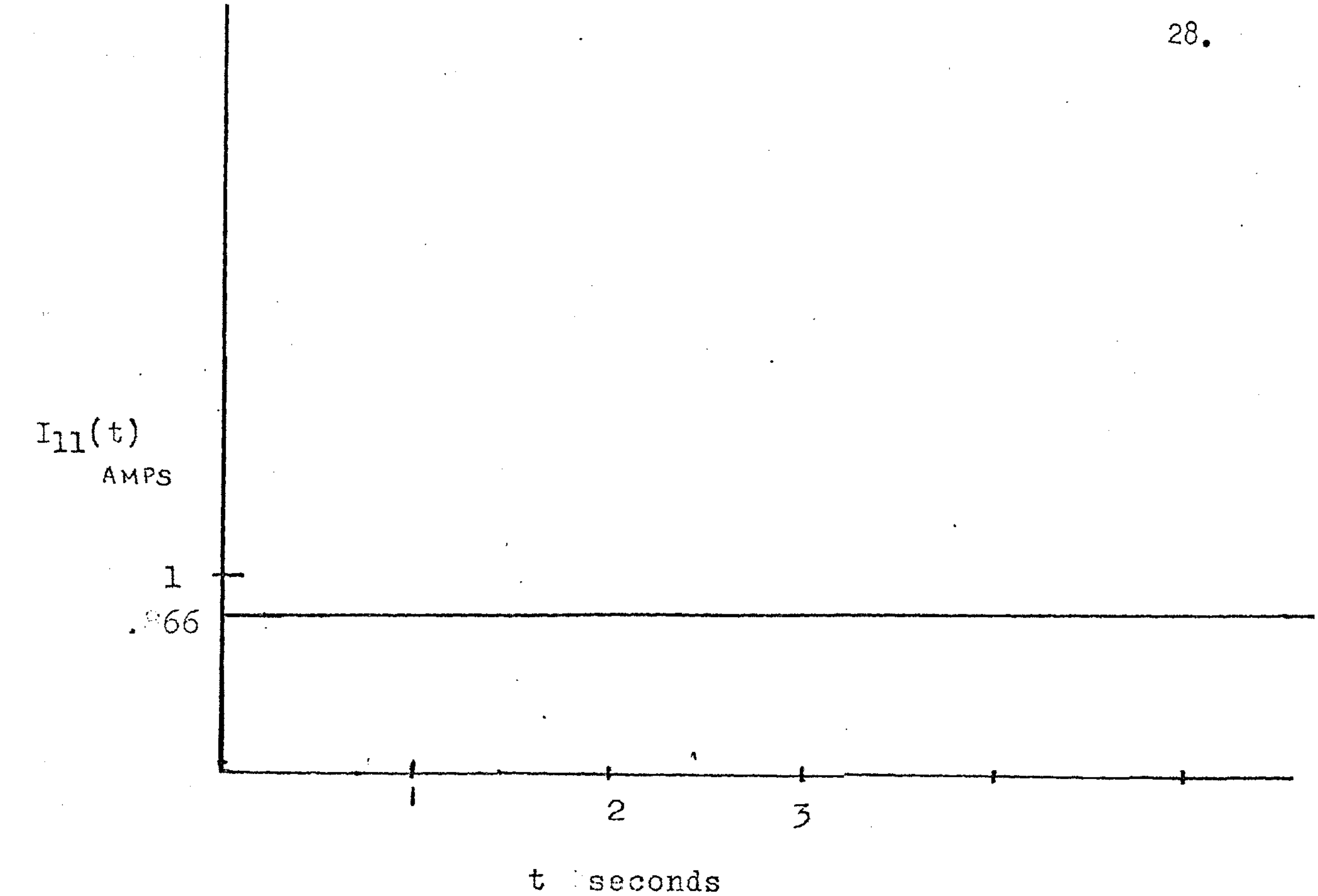


FIG. 10 UNIT STEP RESPONSE OF A 3-WIRE LC LINE, EXAMPLE 1 $a = 1$.



Example 1. "a" = 0

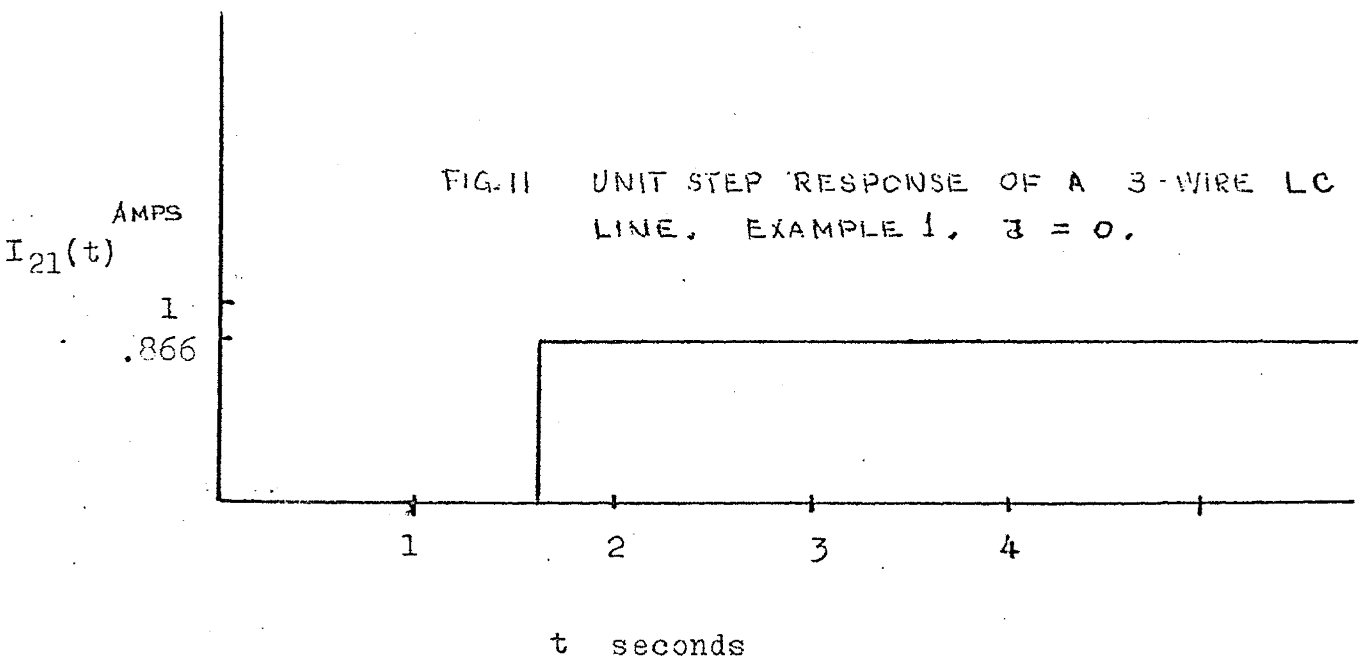


FIG. 11 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 1, a = 0.

Example 1 "a" = 0

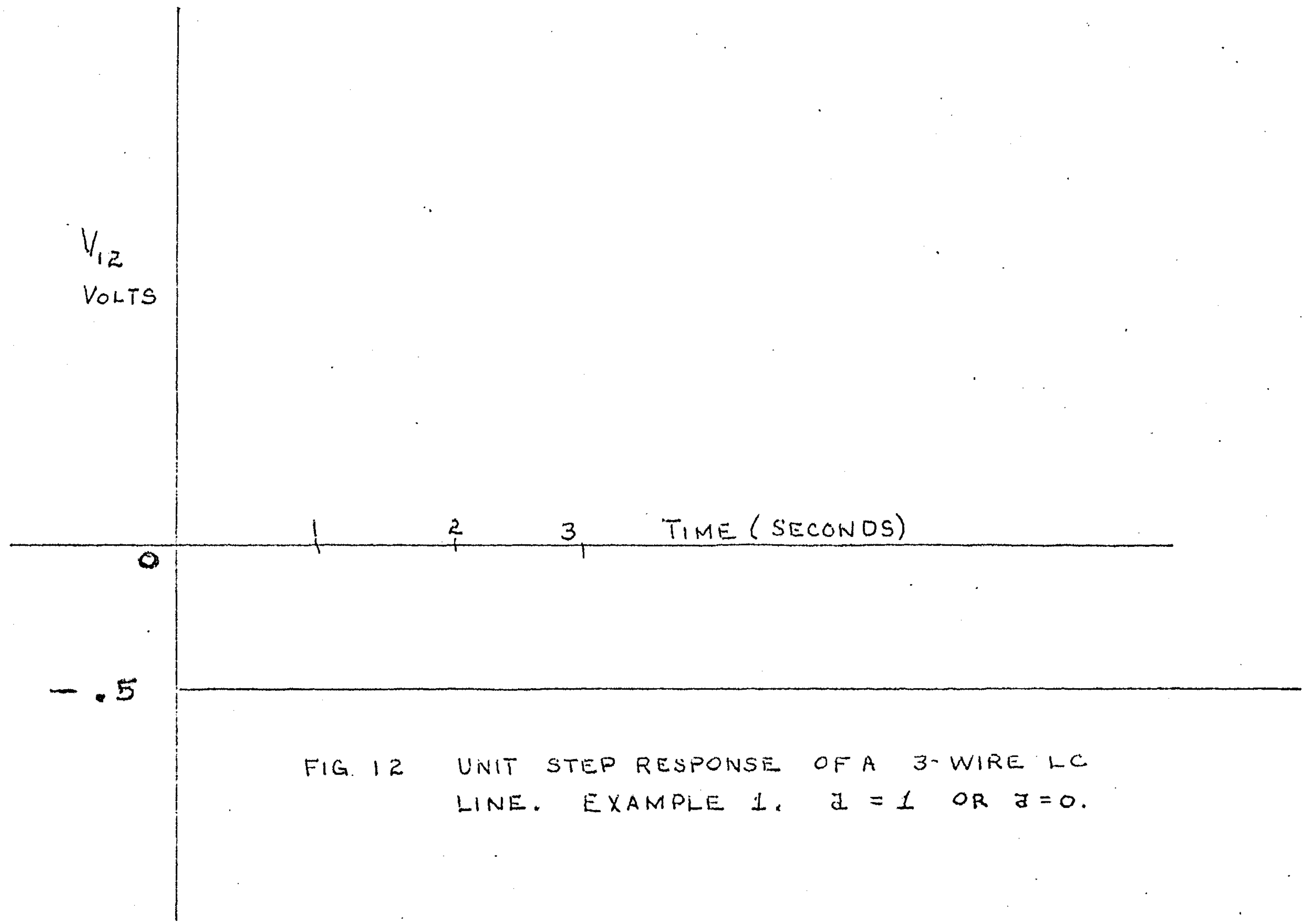
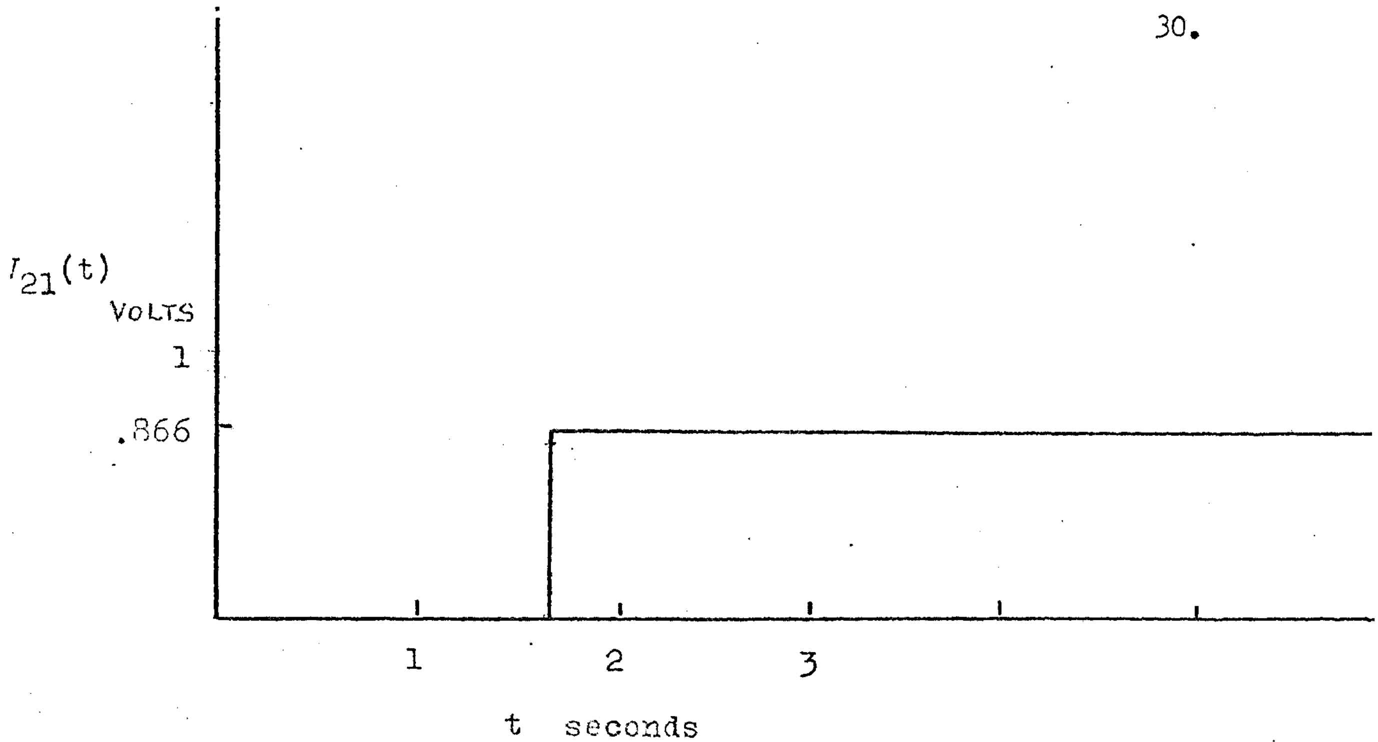


FIG. 12 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 1. $a = 1$ OR $a = 0$.



Example 1. "a" = 0

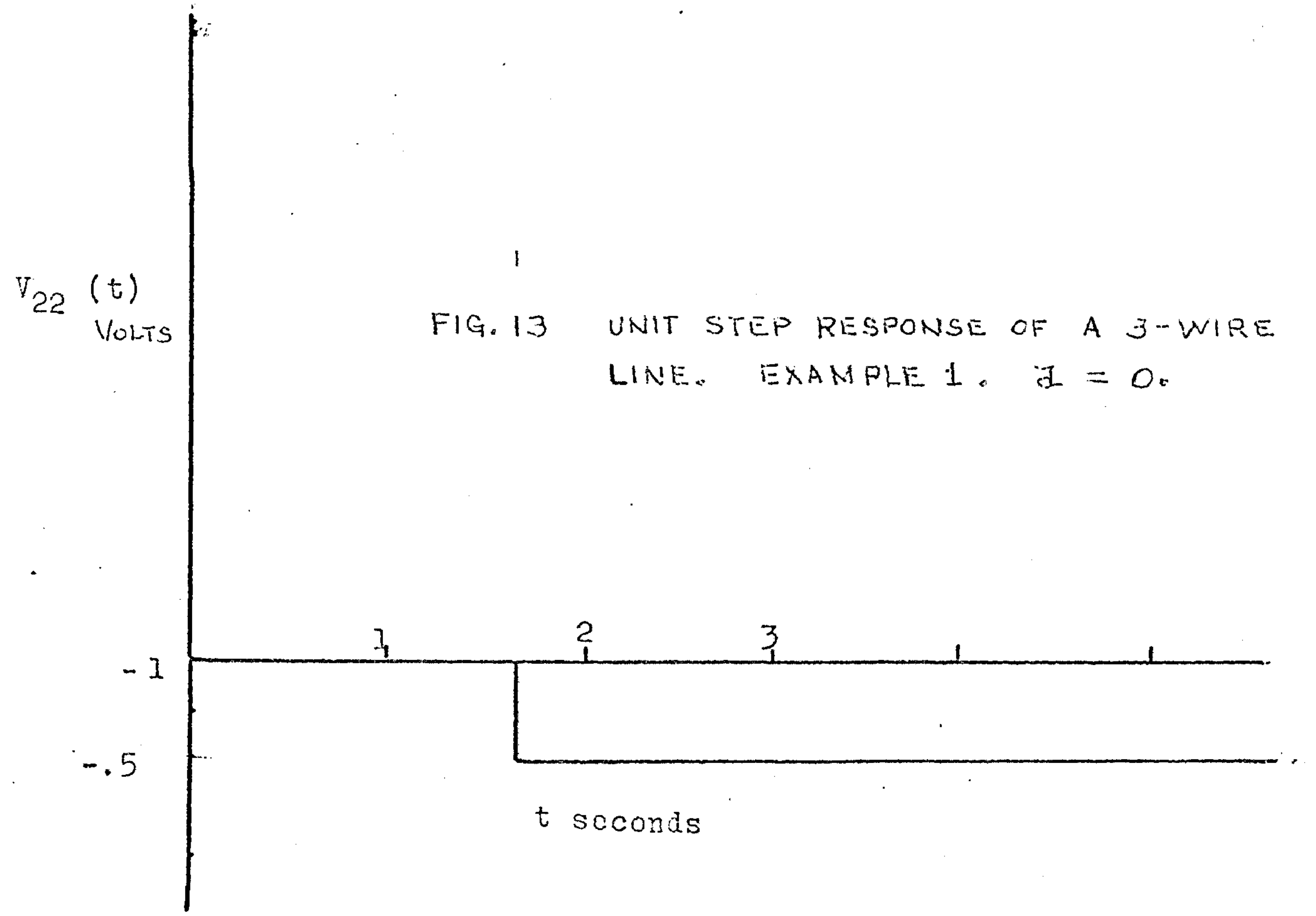
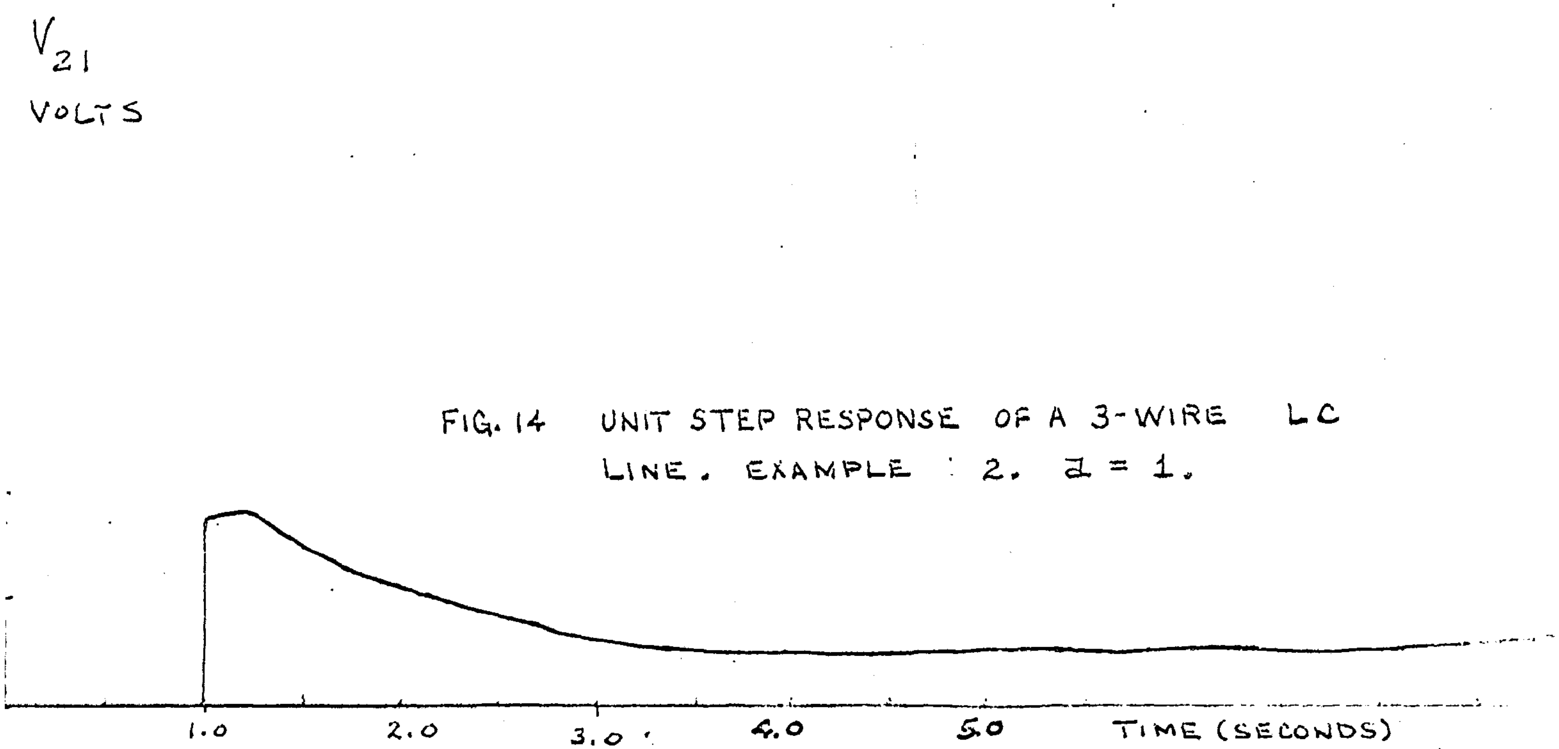


FIG. 13 UNIT STEP RESPONSE OF A 3-WIRE
LINE. EXAMPLE 1. a = 0.

Example 1. "a" = 0

FIG. 14 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 2. $\alpha = 1$.



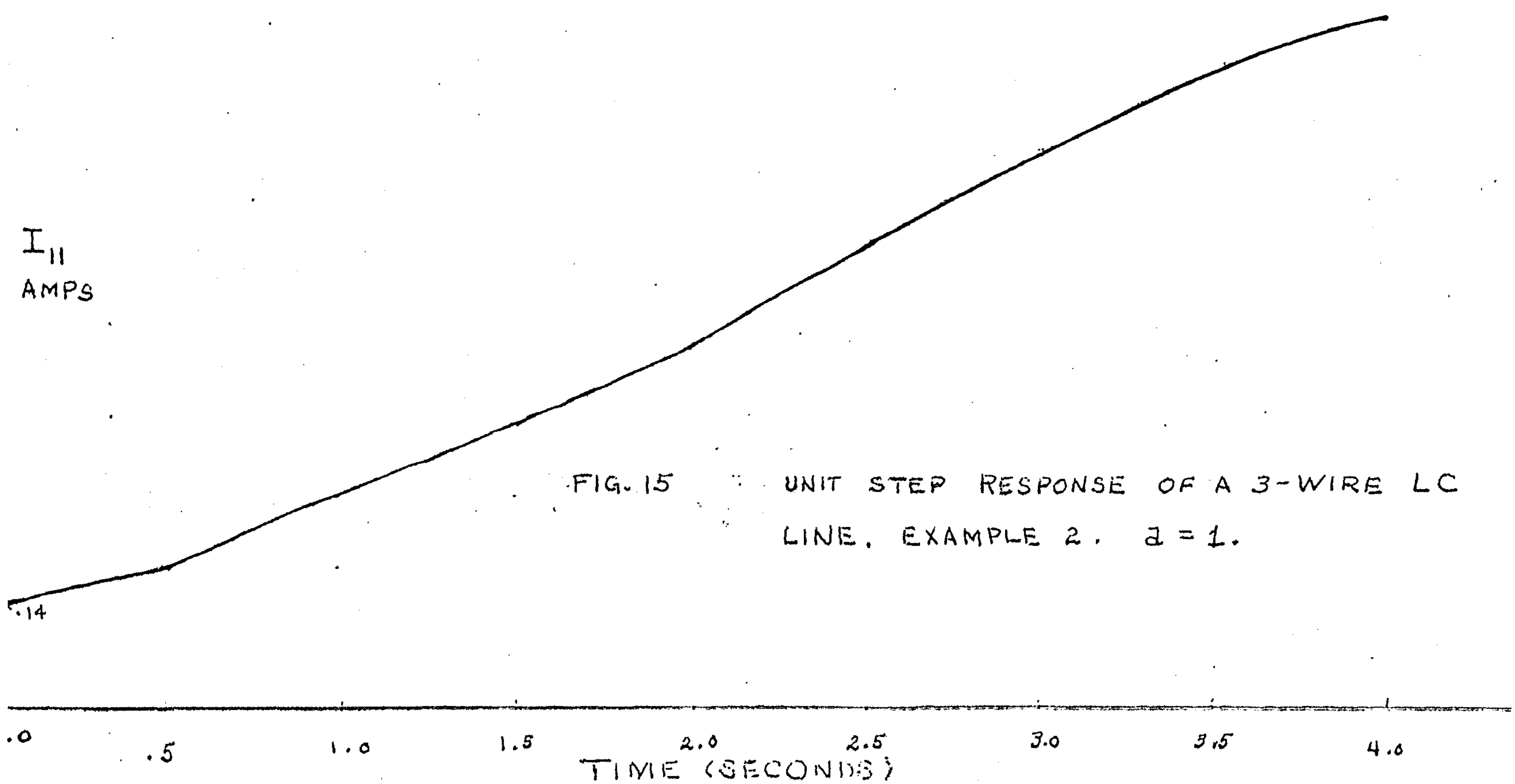


FIG. 15 UNIT STEP RESPONSE OF A 3-WIRE LC LINE, EXAMPLE 2. $a = 1$.

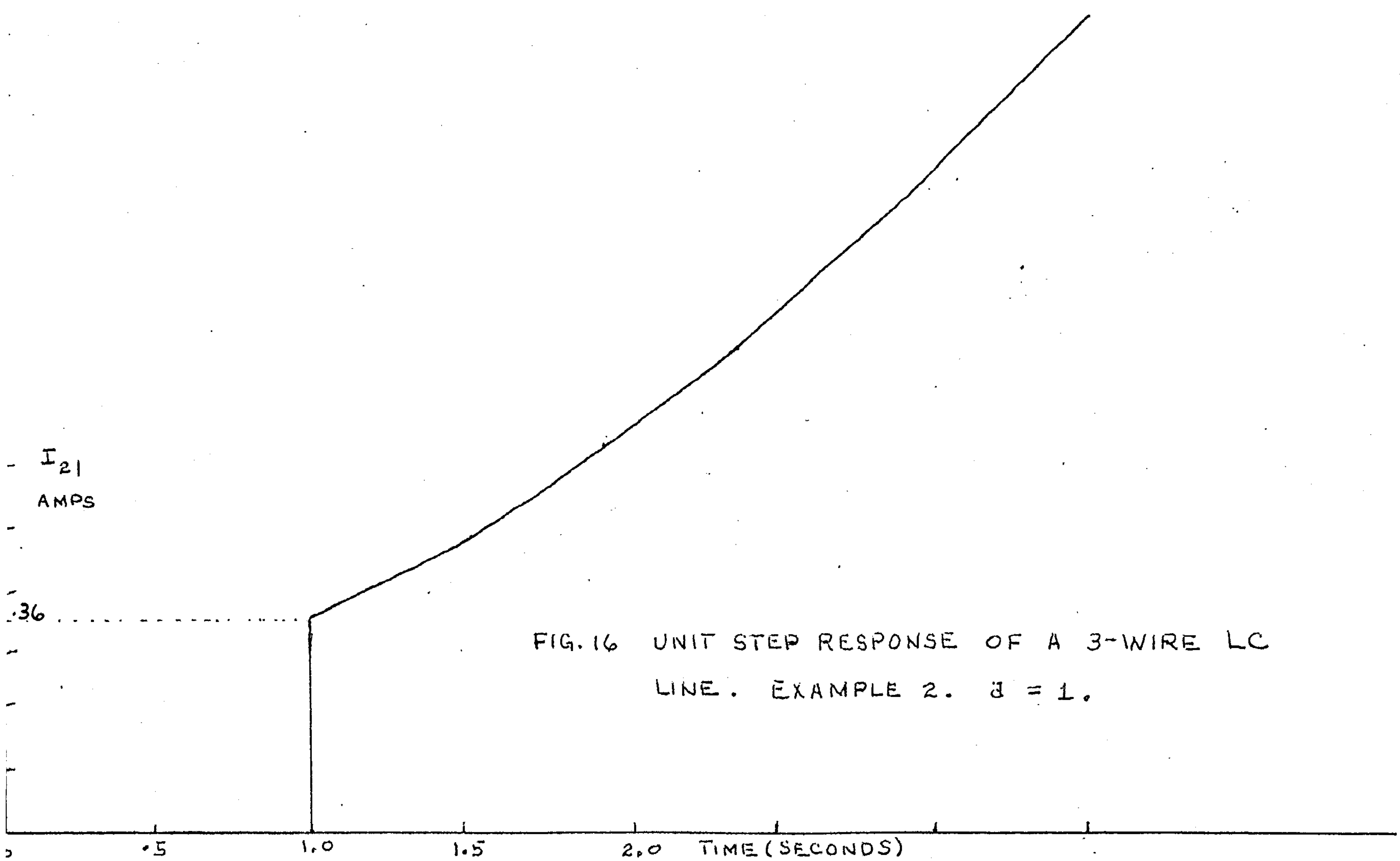
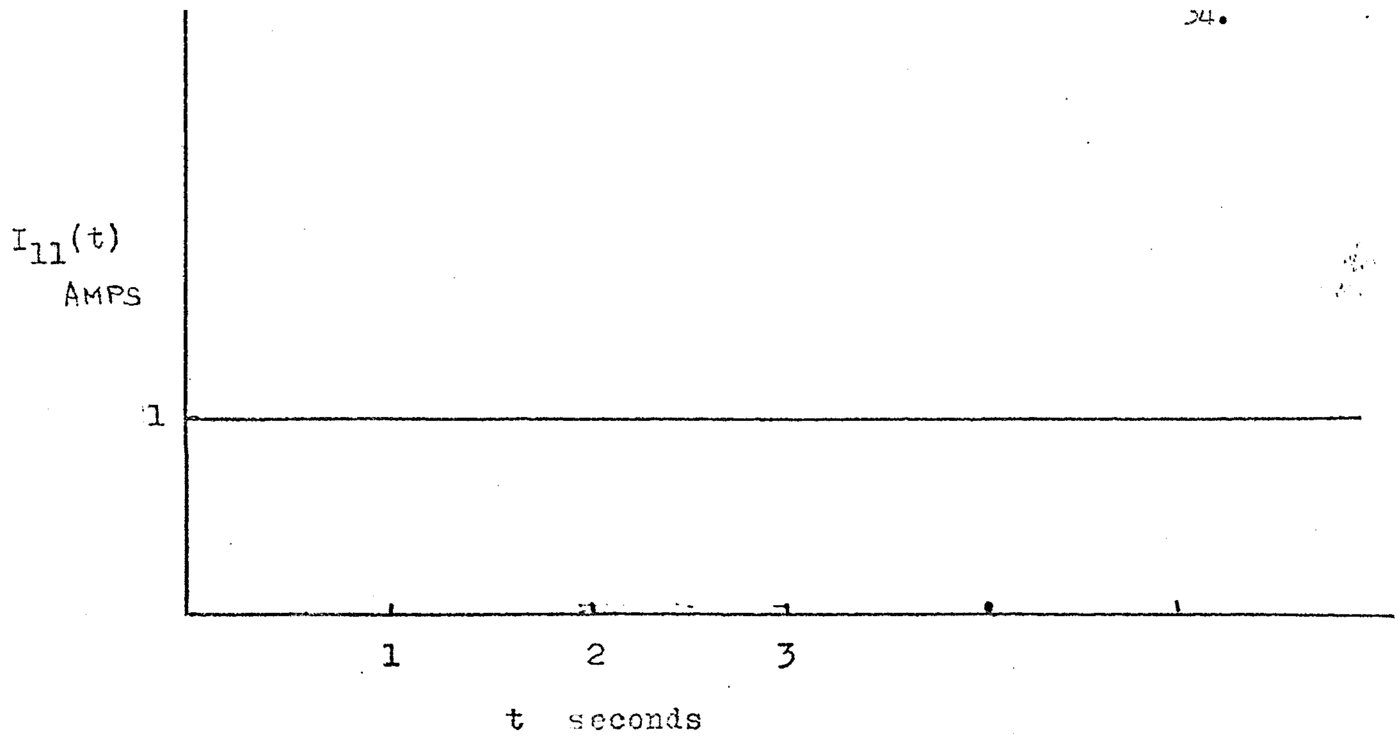


FIG. 16 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 2. $\partial = 1$.



Example 2. "a" = 0

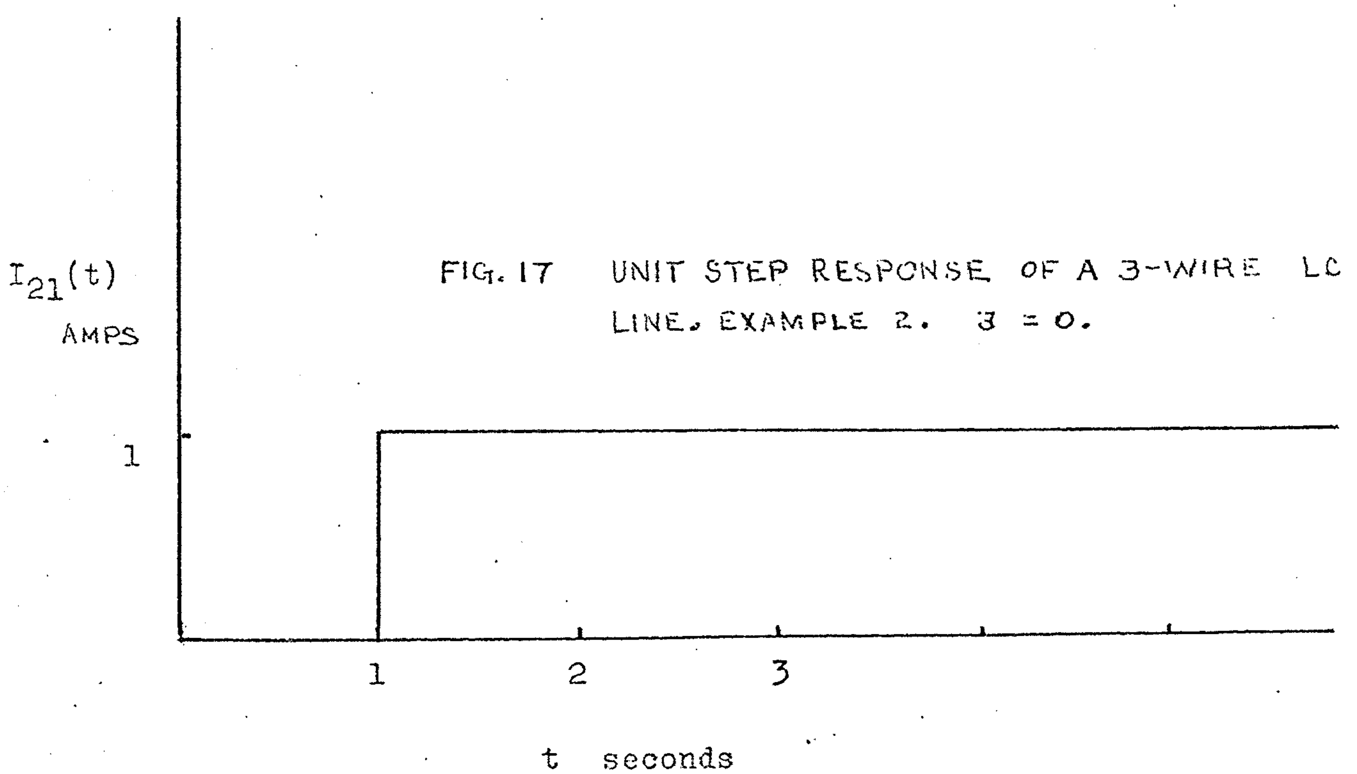


FIG. 17 UNIT STEP RESPONSE OF A 3-WIRE LC LINE. EXAMPLE 2. 3 = 0.

VII. Conclusions

The solutions found in this thesis are for the exponentially tapered transmission line and can be used for a constant parameter line by setting the taper constant "a" equal to zero. Although the examples worked here were for LC lines, the general equations developed in the first part of the thesis apply equally well to any exponential line.

It is apparent that analysis of the problem is much simplified when the line is terminated in the characteristic impedance, since this eliminates the reflected waves of both current and voltage.

It should be noted from the examples, that the current responses increased without bound in the tapered cases, but were bounded for the constant parameter cases. Another important difference between the tapered line and the constant parameter line is that the tapered line acts as a transformer. The responses at the load are not only delayed in time, but the magnitude is also different. When the tapering constant "a" equals zero, this reduces to the constant parameter line case and it is seen that the transformer effect no longer holds between the source and load variables.

VIII. References

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Vita

The author was born on January 19, 1943 in Oak Hill, W. Va. He received his primary and secondary education at Fayetteville, W. Va., and then enrolled in West Virginia Institute of Technology in 1961 at Montgomery, W. Va. He received his Bachelor of Science in Electrical Engineering in June 1966. He has been a full time student at the University of Missouri at Rolla since September 1966.